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ON THE LARGEST PRIME DIVISOR OF AN INTEGER

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On the largest prime divisor of an integer *)

by

J.B. van Rongen **)

ABSTRACT

In this report we consider the number-theoretical sequence $\{\lambda_m\}_{m=1}^{\infty}$, where $\lambda_1 = 1$ and $\lambda_m = \frac{\log m}{\log p(m)}$ $(m \geq 2)$, p(m) being the largest prime divisor of m. For a large class of functions f we derive the average limit $\lim \frac{1}{n} \sum_{m=1}^{n} f(\lambda_m)$.

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0. INTRODUCTION

For integers $m \geq 2$, let $p(m)$ be the largest prime divisor of m, and let λ_m be defined implicitly by $p(m)^{\lambda_m} = m$. It is convenient to take $\lambda_1 = 1$. Recently J. van de Lune ([2]) proposed the following problem. Let $f(x)$ be a function on $[1,\infty)$. Under what conditions on $f(x)$ does

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} f(\lambda_m)$$

exist?

It was shown in [2] that for bounded and continuous functions f, this limit exists and equals

$$-\int_1^\infty f(x)\, d\rho(x)$$

where $\rho(x)$ is Dickman's function defined below. In this note we extend this result to a class of continuous functions f which includes all polynomials. In particular we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \lambda_m = e^\gamma$$

where γ is Euler's constant.

1. SOME AUXILIARY LEMMAS

<u>Lemma 1</u>. *Dickman's function* $\rho(x)$ *is the continuous function defined by the difference-differential equation*

$$\rho'(x) = -\frac{1}{x}\rho(x-1), \qquad (1 < x),$$

with

$$\rho(x) = 1, \qquad (0 \leq x \leq 1).$$

$\rho(x)$ *has the following properties:*

(a) $0 < \rho(\alpha) \le \{\Gamma(\alpha+1)\}^{-1}$, $(\alpha \ge 0)$.

(b) $\rho(\alpha)$ *is non-increasing.*

(c) $\int_0^\infty \rho(\alpha)d\alpha = e^\gamma$ *where* γ *is Euler's constant.*

(d) *For all positive integers* M,k ,

$$\alpha^k\rho^{(k)}(\alpha) = O_{M,k}(\alpha^{-M}), \qquad (\alpha \to \infty) ,$$

where $\rho^{(k)}$ *is the* k*-th derivative of* ρ .

Proof. A proof of lemma 1(a) and (b) can be found in [3, p.27-28]. For (c), see [2]. To prove (d), we first show by induction that rational functions

$$R_{k,j}(\alpha), \qquad k = 1,2,\dots;1 \le j \le k ,$$

exist, such that

$$\alpha^k\rho^{(k)}(\alpha) = \sum_{j=1}^{k} R_{k,j}(\alpha)\rho(\alpha-j), \qquad (\alpha > k) \tag{1.1}$$

and such that $R_{k,j}(\alpha)$ has no poles for $\alpha > k$.

By the definition of ρ, clearly $R_{1,1}(\alpha) = -1$. Suppose we have shown (1.1) for $k \le n$. From

$$\alpha^{n+1}\rho^{(n+1)}(\alpha) = \alpha \frac{d}{d\alpha}\{\alpha^n\rho^{(n)}(\alpha)\} - n\alpha^n\rho^{(n)}(\alpha)$$

we obtain

$$\alpha^{n+1}\rho^{(n+1)}(\alpha) = \alpha \sum_{j=1}^{n}\left[\{\frac{d}{d\alpha} R_{n,j}(\alpha)\}\rho(\alpha-j) - \frac{R_{n,j}(\alpha)}{\alpha-j}\rho(a-j-1)\right] +$$

$$- \sum_{j=1}^{n} n\, R_{n,j}(\alpha)\rho(\alpha-j) .$$

Hence,

$$\alpha^{n+1}\rho^{(n+1)}(\alpha) = \{\alpha R'_{n,1}(\alpha) - n R_{n,1}(\alpha)\}\rho(\alpha-1) +$$

$$+ \sum_{j=2}^{n} \{\alpha R'_{n,j}(\alpha) - n R_{n,j}(\alpha) - \frac{\alpha}{\alpha-j+1} R_{n,j-1}(\alpha)\}\rho(\alpha-j) +$$

$$- \frac{\alpha}{\alpha-n} R_{n,n}(\alpha)\ \rho(\alpha-n-1)\ .$$

Defining $R_{n+1,j}(\alpha)$ in an obvious way, this completes the proof of (1.1). It is also clear that the $R_{n+1,j}$'s do not have poles for $\alpha > n+1$. Noting that $\{\Gamma(\alpha+1)\}^{-1} = O_M(\alpha^{-M})$ for all positive M, and using lemma 1(a) in the RHS of (1.1), the proof is finished. □

We define for $y \geq 2$ and $\alpha > 1$,

$$\Psi(n,y) = \text{card}\{m \in \mathbb{Z} \mid 2 \leq m \leq n;\ p(m) \leq y\}\ ,$$

$$G(n,\alpha) = \text{card}\{m \in \mathbb{Z} \mid 2 \leq m \leq n;\ p(m) \leq m^{1/\alpha}\}.$$

The following two lemmas will give useful estimations for $\Psi(n,n^{1/\alpha})$ and $G(n,\alpha)$.

Lemma 2.

(a) *For* $1 < \alpha \leq (\log n)^{1/2}$ *we have uniformly in* α

$$\Psi(n,n^{1/\alpha}) = n\ \rho(\alpha) + O(\frac{n}{\log n})\ .$$

(b) *Let* a_v, $v=0,1,\ldots$ *be the coefficients in the power series expansion*

$$s(1+s)^{-1}\ \zeta(1+s) = \sum_{v=0}^{\infty} a_v s^v, \qquad |s| < 1\ .$$

Here ζ *is the Riemann* ζ*-function. Let* m *be a positive integer, and suppose* $m < \alpha \leq (\log n)^{1/2}$.

$$\Psi(n,n^{1/\alpha}) = n \sum_{v=0}^{m-1} a_v \alpha^v \rho^{(v)}(\alpha)(\log n)^{-v} + O_m(\frac{\alpha^m n}{(\log n)^m})$$

where $\rho^{(v)}$ *is the* v-*th derivative of* ρ.

Lemma 2(a) is a weakened version of a theorem of Ramaswami [4], see also Norton [3, p.47]. Lemma 2(b) was announced by Ramaswami [4, p.109], but he did not publish a proof. It is an immediate consequence of a theorem of De Bruijn [1].

Lemma 3. *For* $1 < \alpha \le (\log n)^{1/2} \cdot (1 - \frac{\log\log n}{\log n})$ *we have uniformly in* α

$$G(n,\alpha) = n\,\rho(\alpha) + O(\frac{n \log\log n}{\log n}) .$$

Proof. Let $2 \le n_1 < n$, then

$$(1.2)\qquad \begin{aligned} \Psi(n,n_1^{1/\alpha}) - \Psi(n_1,n_1^{1/\alpha}) &= \mathrm{card}\{m \in \mathbb{Z} \mid n_1<m\le n;\ p(m)\le n_1^{1/\alpha}\} \le \\ &\le \mathrm{card}\{m \in \mathbb{Z} \mid n_1<m\le n;\ p(m)<m^{1/\alpha}\} \le \\ &\le \mathrm{card}\{m \in \mathbb{Z} \mid 2\le m\le n;\ p(m)\le m^{1/\alpha}\} = \\ &= G(n,\alpha) . \end{aligned}$$

It is obvious that

$$(1.3)\qquad G(n,\alpha) \le \Psi(n,n^{1/\alpha}) .$$

Suppose that $1 < \alpha \le (\log n)^{1/2}(1 - \log\log n/\log n)$. We take $n_1 = n(\log n)^{-1}$, $\beta = \alpha(1 - \log\log n/\log n)^{-1}$. Hence

$$(1.4)\qquad n_1^{1/\alpha} = n^{1/\beta} \quad \text{and} \quad \beta \le (\log n)^{1/2}.$$

According to lemma 2(a), there is an absolute constant K, such that for $1 < \alpha \le (\log n)^{1/2}$

$$(1.5)\qquad |\Psi(n,n^{1/\alpha}) - n\,\rho(\alpha)| \le K \frac{n}{\log n} .$$

Hence, using the trivial estimate $\Psi(n_1,n_1^{1/\alpha}) \le n_1 = n(\log n)^{-1}$ in (1.2), we have

$$(1.6) \qquad G(n,\alpha) \ge \Psi(n,n^{1/\beta})-\Psi(n_1,n_1^{1/\alpha}) \ge n\,\rho(\beta)-\frac{(K+1)n}{\log n}\ .$$

On the other hand, (1.3) and (1.5) immediately give

$$(1.7) \qquad G(n,\alpha) \le n\,\rho(\alpha)+K\frac{n}{\log n}\ ;$$

Finally, we estimate $\rho(\alpha) - \rho(\beta)$. From the definition of ρ, lemma 1(a) and (b) we have

$$0 < \rho(\alpha) - \rho(\beta) = \int_\alpha^\beta t^{-1}\rho(t-1)dt \le \frac{\beta-\alpha}{\alpha}\,\rho(\alpha-1) \le \frac{\beta}{\alpha} - 1 =$$
$$= O(\mathrm{loglog}\ n/\log n)\ .$$

A combination of (1.6) and (1.7) now proves the lemma. □

2. MAIN RESULT

Theorem. *Let* f(x) *be a continuous and monotonic function of* x *on* $[1,\infty)$, *such that a positive integer* N *exists with*

$$f(x) = O_N(x^N), \qquad (x \to \infty).$$

Then

$$\frac{1}{n}\sum_{m=1}^{n} f(\lambda_m) = -\int_1^\infty f(\alpha)d\rho(\alpha) + O_N(\mathrm{loglog}\ n/(\log n)^{1/(N+1)})\ .$$

Proof. $G(n,\alpha)$ is already defined for $\alpha > 1$. For $0 < \alpha \le 1$ we define $G(n,\alpha) = [n]$. Fix $n > 2$. $G(n,\alpha)$ is a left-continuous stepfunction of α, with a finite number of jumps, say at $1=\alpha_1<\alpha_2<\ldots<\alpha_\nu$. Clearly $G(n,\alpha) = 0$ for $\alpha > \alpha_\nu$.

Define the characteristic functions $\chi(\alpha,m)$ for $\alpha > 0$, $m \geq 2$ by

$$\chi(\alpha,m) = \begin{cases} 1 & \text{if } p(m) \leq m^{1/\alpha}, \\ 0 & \text{if } p(m) > m^{1/\alpha}. \end{cases} \tag{2.1}$$

Furthermore, $\chi(\alpha,1) = 1$ if $0 < \alpha \leq 1$, $\chi(\alpha,1) = 0$ elsewhere. Take $\alpha_0 = 0$, $f(0) = 0$. Then we have:

$$\begin{aligned} \sum_{m=1}^{n} f(\lambda_m) &= \sum_{m=1}^{n} \sum_{k=1}^{\nu} \{f(\alpha_k)-f(\alpha_{k-1})\}\chi(\alpha_k,m) = \\ &= \sum_{k=1}^{\nu} \{f(\alpha_k)-f(\alpha_{k-1})\} \sum_{m=1}^{n} \chi(\alpha_k,m) = \\ &= \sum_{k=1}^{\nu} \{f(\alpha_k)-f(\alpha_{k-1})\}G(n,\alpha_k) = \\ &= \int_1^{\infty} G(n,\alpha)df(\alpha) + f(1)G(n,1) . \end{aligned} \tag{2.2}$$

Here the integral and all following integrals are Riemann-Stieltjes integrals. It is easy to see that $\alpha_\nu \leq \log n/\log 2 < 2 \log n$. Therefore, instead of (2.2) we may write as well

$$\sum_{m=1}^{n} f(\lambda_m) = \int_1^{2 \log n} G(n,\alpha)df(\alpha) + f(1)G(n,1) . \tag{2.3}$$

Suppose that $f(x) = O_N(x^N)$, $(x \to \infty)$, for some $N \geq 2$. We split the above integral into two parts:

$$\int_1^{2 \log n} = \int_1^{z} + \int_z^{2 \log n} . \tag{2.4}$$

Here $z = (\log n)^{1/(N+1)}$. For the first integral we have, according to lemma 3 and the monotonicity of f:

$$\int_1^z G(n,\alpha)df(\alpha) = n\int_1^z \{\rho(\alpha) + O\left(\frac{\text{loglog } n}{\log n}\right)\}df(\alpha) =$$

$$(2.5) \qquad = n\int_1^z \rho(\alpha)df(\alpha) + O\left(\frac{n \text{ loglog } n}{\log n}\int_1^z |df(\alpha)|\right) =$$

$$= n\int_1^z \rho(\alpha)df(\alpha) + O_N\left(\frac{n \text{ loglog } n}{(\log n)^{1/(N+1)}}\right) .$$

By the definition of ρ, $f(1)G(n,1) = n\, f(1)\rho(1) + O(1)$. Furthermore, using lemma 1(a) we have

$$f(z)\rho(z) = O_N\left((\log n)^{N/(N+1)} \cdot \{\Gamma((\log n)^{1/(N+1)})\}^{-1}\right) =$$

$$= O_N((\log n)^{-1}) .$$

We also have, using the same estimate

$$\int_z^\infty f(\alpha)d\rho(\alpha) = O_N(-\int_z^\infty \alpha^N\, d\rho(\alpha)) = O_N((\log n)^{-1}) .$$

Hence, by partial integration we have

$$(2.6) \qquad n\int_1^z \rho(\alpha)df(\alpha) = -\, f(1)G(n,1) - n\int_1^z f(\alpha)d\rho(\alpha) + O_N(\frac{n}{\log n}) =$$

$$= -\, f(1)G(n,1) - n\int_1^\infty f(\alpha)d\rho(\alpha) + O_N(\frac{n}{\log n}) .$$

Combining (2.5) and (2.6) we get

$$(2.7)\qquad \int_1^z G(n,\alpha)df(\alpha) = - f(1)G(n,1) - n\int_1^\infty f(\alpha)d\rho(\alpha) + $$

$$+ O_N\left(\frac{n \log\log n}{(\log n)^{1/(N+1)}}\right) .$$

It remains to show that the second integral in the RHS of (2.4) is small. As $\Psi(n,n^{1/\alpha})$, for fixed n, is a non-increasing function of α, we have for $\alpha \geq z$:

$$G(n,\alpha) \leq \Psi(n,n^{1/\alpha}) \leq \Psi(n,n^{1/z}) .$$

Suppose that $\alpha \geq z > N+2$ (i.e. $n > \exp\{(N+2)^{N+1}\}$). Using lemma 2(b) we have

$$(2.8)\qquad G(n,\alpha) \leq n\sum_{v=0}^{N+1} a_v\{z^v\rho^{(v)}(z)\}(\log n)^{-v} + O_N\left(\frac{n}{(\log n)^{N+1-\frac{1}{N+1}}}\right) .$$

According to lemma 1(d), substituting $M = (N+1)(N+1-v)$

$$z^v\rho^{(v)}(z) = O_M(z^{-M}) = O_N((\log n)^{-N-1+v}) .$$

Hence from (2.8) we can conclude

$$G(n,\alpha) = O_N\left(\frac{n}{(\log n)^{N+1-\frac{1}{N+1}}}\right)$$

for $\alpha \geq (\log n)^{1/N+1}$. Therefore we have the following estimate:

$$(2.9)\qquad \int_{z}^{2\log n} G(n,\alpha)df(\alpha) = O_N\left(\frac{n}{(\log n)^{N+1-\frac{1}{N+1}}} \cdot \int_{z}^{2\log n} |df(\alpha)|\right) =$$

$$= O_N\left(n\ (\log n)^{\frac{-N}{N+1}}\right).$$

Combining (2.3), (2.4), (2.7) and (2.9), the proof is complete. □

Corollary 1. The theorem is also valid for functions, which are the difference of two monotonic functions, both of order $O_N(x^N)$ for some N. In particular, it holds for all polynomials.

Corollary 2. $\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \lambda_m = e^{\gamma}$.

This result can be arrived at by taking $f(\alpha) = \alpha$. The integral then equals, by partial integration

$$-\int_{1}^{\infty} \alpha\ d\rho(\alpha) = -\int_{0}^{\infty} \alpha\ d\rho(\alpha) = \int_{0}^{\infty} \rho(\alpha)d\alpha = e^{\gamma}$$

according to lemma 1(c).

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